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*Phil. Trans. R. Soc. Lond. A* 1970 **266**, 1-33
doi: 10.1098/rsta.1970.0001

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# FRICTIONAL TRACTION AND LUBRICANT RHEOLOGY IN ELASTOHYDRODYNAMIC LUBRICATION

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(*Communicated by Sir Derman Christopherson, F.R.S.—Received* 9 *May* 1969)

## CONTENTS

## Notation

Subscripts M and BL refer to Maxwell and to Barlow–Lamb liquids respectively.
Superscript * denotes complex quantity.
Superscripts ′ and ″ refer to real and imaginary parts respectively of a complex quantity, e.g. $G^*(\mathrm{i}\omega) = G'(\omega)+\mathrm{i}G''(\omega)$.

| | |
|---|---|
| $A^* = A'-\mathrm{i}A''$ | complex mechanical admittance (admittance = reciprocal of impedance) in oscillatory shear |
| $b$ | half width of Hertzian conjunction region between two disks |
| $D$ | shear rate |
| $f$ | coefficient of frictional traction |
| $f_{\max}$ | maximum coefficient of frictional traction |
| $F$ | frictional traction |
| $G^* = G'+\mathrm{i}G''$ | complex modulus in oscillatory shear |
| $G_\infty$ | limiting shear modulus in oscillatory shear at very high frequencies |
| $h_0$ | minimum thickness of lubricant film, assumed to be approximately constant in Hertzian conjunction region |
| i | operator denoting out-of-phase component in oscillatory phenomena |
| $J^* = J'-\mathrm{i}J''$ | complex compliance (= reciprocal of modulus) in oscillatory shear |
| $k_0$ | thermal conductivity of lubricant |
| $K$ | arbitrary shift constant in relation between behaviour of viscoelastic liquid in oscillatory and in continuous shear, $\omega = KD$ |
| $L^{-1}$ | inverse Laplace transform |
| $N$ | normal load on disks |
| $p$ | pressure |
| $\bar{p}$ | mean Hertzian pressure in conjunction between two disks |
| $s = \mathrm{i}\omega$ | |
| $t$ | present time |
| $t_1 = 2b/\bar{U}$ | mean time of exposure of liquid to shear field |
| $T$ | absolute temperature |
| $u$ | past time |
| $U_1, U_2$ | peripheral velocities of disks relative to their conjunction |
| $\bar{U} = \frac{1}{2}(U_1+U_2)$ | mean rolling velocity |
| $v = t-u$ | |
| $Z^* = Z'+\mathrm{i}Z''$ | complex mechanical impedance in oscillatory shear |
| $\alpha$ | pressure coefficient of viscosity |
| $\alpha_1, \alpha_2$; $\beta, \beta_1, \beta_2$ | constants in relation between viscosity and temperature |
| $\gamma$ | shear strain |

| | |
|---|---|
| $\Delta = \eta_0 D/G_\infty$ | dimensionless shear rate |
| $\zeta = G_\infty t/\eta_0$ | dimensionless time |
| $\zeta_M$ | $t/\lambda_M$ |
| $\zeta_1$ | $G_\infty t_1/\eta_0$ |
| $\zeta_{M1}$ | $t_1/\lambda_M$ |
| $\eta$ | viscosity |
| $\eta_0$ | true viscosity, viscosity at zero rolling speed |
| $\eta_a$ | apparent viscosity |
| $\eta_w$ | viscosity at temperature $\theta_w$ of walls bounding film of lubricant |
| $\eta^* = \eta' - i\eta''$ | complex dynamic viscosity in oscillatory shear |
| $\theta$ | temperature |
| $\bar{\theta}_c$ | mean temperature in median plane of film of lubricant |
| $\theta_w$ | temperature of boundaries of film of lubricant |
| $\lambda = \eta_0/G_\infty$ | relaxation time |
| $\xi = \tau/\eta_0 D$ | dimensionless shear stress |
| $\bar{\xi}$ | $\bar{\tau}/\eta_0 D$ |
| $\rho$ | density of liquid |
| $\tau$ | shear stress |
| $\tau_\infty$ | equilibrium value of shear stress |
| $\bar{\tau}$ | mean shear stress over conjunction region |
| $\phi^* = \phi' + i\phi''$ | fluidity (reciprocal of viscosity) in oscillatory shear |
| $\chi_0$ | thermal diffusivity of lubricant |
| $\psi$ | $\eta_w \beta (U_1 - U_2)^2/8k_0$ |
| $\bar{\psi}$ | relaxation function |
| $\omega$ | angular frequency in oscillatory shear |
| $\Omega = \eta_0 \omega/G_\infty$ | dimensionless frequency in oscillatory shear |

Much experimental information on frictional traction in elastohydrodynamic lubrication has been published, but there is no adequate theory to explain the many puzzling features of the results. In this paper, some of the friction results are interpreted in terms of a recently published model of a viscoelastic liquid. This model refers to behaviour in oscillatory shear, and in this work the implications for the behaviour in continuous shear have been examined.

The experimental curves of frictional traction against sliding speed have been analysed into three regions; the linear region, the nonlinear (ascending) region, and the thermal (descending) region.

In the linear region there is a necessary and close connexion between the friction and the behaviour in oscillatory shear, and many of the experimental features observed have been explained in terms of the new viscoelastic model, which is considerably superior to the more conventional Maxwell model. Some of the experimental results are still unexplained.

In the nonlinear (ascending) region, the relation between results in oscillatory shear and those in continuous shear is far less clear-cut, but it is possible to show some measure of agreement between the two fields.

In the thermal (descending) region, some of the precision is regained, and an empirical relation proposed in the analysis of the nonlinear (ascending) region is found to give a good quantitative agreement with published results.

## Introduction

The formation of the film of lubricant necessary for the correct functioning of heavily loaded bearing surfaces, such as those of gears, cams and tappets, etc., depends on the mechanism known as elastohydrodynamic lubrication. The loaded surfaces may fail by scuffing or scoring and this failure is difficult to predict or to understand. Two elements essential to an understanding of this form of failure are the values of the minimum film thickness and the coefficient of sliding friction, and these quantities are also of great practical interest in themselves.

An adequate theory for the minimum film thickness is available (Dowson & Higginson 1961; Dowson, Higginson & Whittaker 1962) and has been extensively tested against experiment. Good agreement was obtained over a wide range of lubricants and of experimental conditions (Dyson, Naylor & Wilson 1965–6) but some discrepancies were noted. These discrepancies have been explained in principle, and suitable modifications to the basic theory have resulted in reasonable agreement with experiment (Dyson & Wilson 1965–6, 1969).

The position as regards frictional traction is much less favourable, and there is no adequate quantitative theory that will predict or explain the experimental results. In the theory for the film thickness, it was assumed that the lubricant was an isothermal Newtonian liquid, with a viscosity increasing exponentially with pressure. A few elementary calculations show that thermal effects in the lubricant film must be very important in the range of sliding speeds of practical interest, but both Smith (1960, 1962) and Crook (1963) pointed out that thermal effects in a Newtonian lubricant were insufficient to explain their experimental results. Crook (1963) suggested that non-Newtonian behaviour was involved, while Smith (1960, 1962) went so far as to postulate a limiting shear strength at which the fluid film failed in a manner similar to that of a plastic solid.

The main difficulty encountered in any theoretical treatment is that of obtaining information from other fields as to the behaviour to be expected from a lubricant under the conditions of elastohydrodynamic lubrication. These conditions are severe and unusual: shear rates range up to $10^7\,s^{-1}$, temperatures up to 670 K (400 °C), pressures up to several $GN\,m^{-2}$ (*ca.* $10^{10}$ dyn $cm^{-2}$; 145 000 $lbf\,in^{-2}$), and the duration of exposure to these conditions is of the order of 1 to 100 $\mu$s. It is impossible to reproduce this situation in any controlled rheological experiment, and indeed it seems impossible to reproduce it at all outside an elastohydrodynamic contact.

In an attempt to circumvent this difficulty, it seems reasonable to attribute the non-Newtonian behaviour of lubricants to viscoelasticity, and to make use of the analogy between the behaviour of a viscoelastic liquid in oscillatory and in continuous shear. An attempt to interpret the results of Smith and of Crook on these lines has already been made (Dyson 1965), information about the behaviour of mineral oils in oscillatory shear being obtained from the results of Barlow & Lamb (1959). There were some unsatisfactory features in this treatment, including unexplained experimental effects, the principal such effect being the variation with rolling speed of the effective viscosity at low sliding speeds, first reported by Crook (1963).

Further very important information about the frictional traction in elastohydrodynamic lubrication has recently been published (Johnson & Cameron 1967–8; Plint 1967–8), and in the meantime a notable advance in an understanding of the viscoelastic properties of ordinary liquids has been made by Barlow, Lamb and co-workers who have proposed a new model of a viscoelastic liquid (Barlow, Lamb, Matheson, Padmini & Richter 1967; Barlow, Erginsav & Lamb 1967). The time therefore seems opportune for a re-examination of the relation between frictional traction in elastohydrodynamic lubrication and the viscoelastic properties of the lubricant.

This paper will deal first with the main features of the experimental results concerning frictional traction. Then the new model of the viscoelastic behaviour of ordinary liquids, due to Barlow, Lamb and co-workers, will be explained, and the relation between these two fields of work will be explored.

## Outline of experimental results in frictional traction

### *The experimental situation*

Most of the useful experimental work in frictional traction in elastohydrodynamic lubrication has been done with disk machines. Two disks, usually of hardened steel, are loaded together and rotated in the presence of a lubricant. The situation is illustrated in figure 1, where the surfaces of the disks have velocities $U_1$ and $U_2$ relative to their conjunction. The disks deform elastically to give a conjunction region $A_1B_1A_2B_2$, of width $2b$ approximately equal to that in the corresponding Hertzian case of dry contact. The surfaces are approximately parallel in this region; the approximately constant film thickness, $h_0$, may be obtained from isothermal theory (Dowson & Higginson 1961; Dowson, Higginson & Whittaker 1962) in most cases. It should be noted that $h_0 \sim 0.1-1\,\mu$m and $b \sim 0.1$ mm, the width $A_1B_1 = A_2B_2$ of the conjunction being much greater than the separation $A_1A_2 = B_1B_2$.

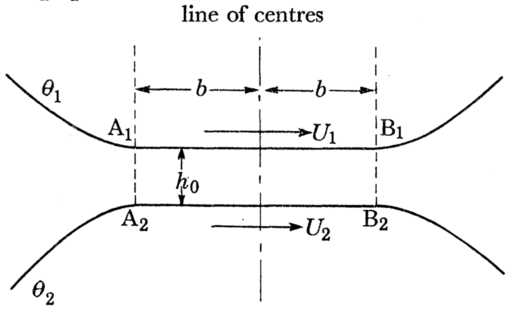


FIGURE 1. The conjunction region in a disk machine.

In a typical experiment, the temperatures $\theta_1$ and $\theta_2$ of the incoming surfaces and the mean rolling velocity $\bar{U} = \frac{1}{2}(U_1+U_2)$ are kept constant. This means that the film thickness $h_0$ in the conjunction is also constant. The frictional traction, $F$, is measured as a function of the sliding speed $(U_1-U_2)$. If all the frictional traction is assumed to arise from shearing of the lubricant in the conjunction, then since the film thickness is constant this relation between frictional traction and sliding speed is of the same form as the relation between the mean shear stress and the mean shear rate in the conjunction. The features of individual curves of traction against sliding speed will be discussed first and then the changes in these main features as the experimental conditions are varied.

### *Features of curves of traction against sliding speed*

Apart from a very small rolling component of friction, the typical curve of traction against sliding speed, illustrated in figure 2, may be divided into three main regions.

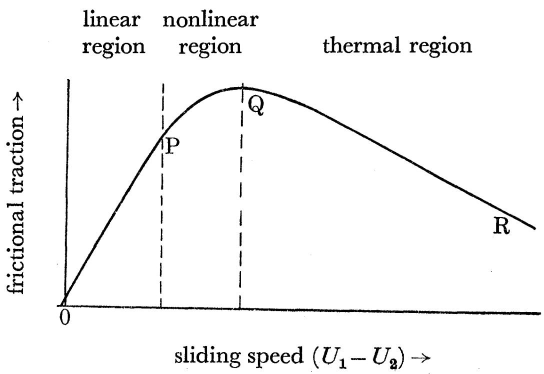


FIGURE 2. Variation of frictional traction with sliding speed.

At low sliding speeds the relation is linear. In view of the close connexion between the experimental curve and the shear stress-shear rate relation, mentioned previously, this may be called the quasi-Newtonian region, with a constant viscosity defined by the slope.

The linear portion of the curve in figure 2 is succeeded by a nonlinear region PQ rising to a maximum at Q. As the sliding speed increases, variations of temperature within the oil film became more and more important. The maximum rise of temperature in the oil film, compared with the temperatures of the bounding surfaces, may be calculated very simply (Archard 1958/9). Such calculations applied to published experimental results show that this rise in temperature is usually between 5 and 10 K at the friction maximum. Thermal effects are therefore relatively unimportant below the maximum, in the region PQ, but they are quite important, and may even be dominant, in the region QR beyond the maximum. This region will therefore be called the thermal region.

*Variation with conditions of features of the friction curves*

The three prominent features that characterize individual curves of frictional traction against sliding speed are the slope of the initial linear portion, the maximum value of the coefficient of friction, and the general agreement between the shapes of the curves in the thermal region. The variation with experimental conditions of each of these three features will now be discussed.

The slope of the linear portion defines a viscosity, which might be expected to remain constant at constant load and temperature. Crook (1963) pointed out that this viscosity decreased with increasing rolling speed, and proposed an explanation in terms of a Maxwell viscoelastic liquid. But he had to assume values of the shear modulus, $G_\infty$, as low as 5 MN m$^{-2}$ ($5 \times 10^7$ dyn cm$^{-2}$), whereas the experiments of Barlow & Lamb (1959) in oscillatory shear gave values approaching 1 GN m$^{-2}$ ($10^{10}$ dyn cm$^{-2}$). This value may require modification in the light of subsequent work in which the modulus was found to decrease with increasing temperature (Barlow, Erginsav & Lamb 1967) and to increase with increasing pressure (Pursley 1968), but the value could scarcely be put at less than 100 MN m$^{-2}$ ($10^9$ dyn cm$^{-2}$). Furthermore, it has been shown (Dyson

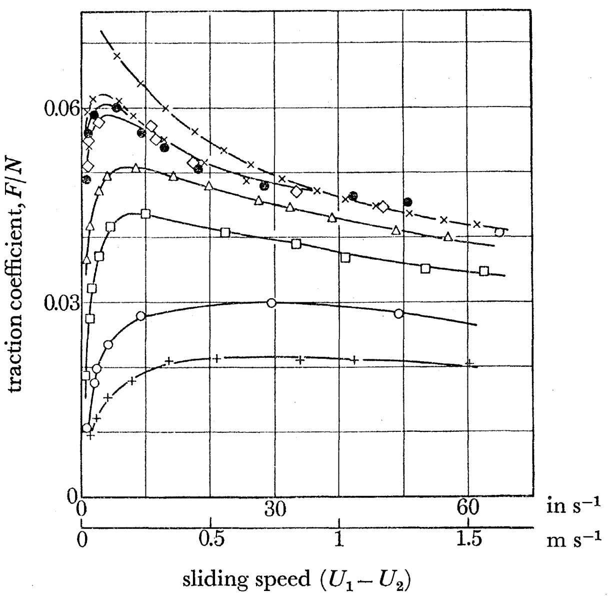


FIGURE 3. Variation of traction with sliding speed at various contact pressures (from Johnson & Cameron 1967–8): maximum Hertz pressure, GN m$^{-2}$ ($10^3$ lbf in$^{-2}$): +, 0.43 (62); ○, 0.60 (87); □, 0.76 (110); △, 1.01 (147); ◇, 1.21 (176); ●, 1.38 (200); ×, 1.55 (224); ▽, 1.67 (243).

1965) that Crook's explanation in terms of the Maxwell model was inconsistent with the observed extent of the linear portion of the graph of traction against sliding speed. A tentative explanation in terms of the Lamb–Barlow model of a viscoelastic liquid will be advanced later.

The effect of the normal load $N$ on the curves of traction against sliding speed is shown in figure 3, reproduced from Johnson & Cameron (1967–8). Particular attention is directed to the variation with pressure of the maximum value of the coefficient of frictional traction, $F/N$. At low pressure this increases sharply with pressure and it is shown in appendix A that this is typical of the behaviour of a Newtonian lubricant which is dominated by thermal effects. The treatment is very much simplified, but a qualitatively similar conclusion is unavoidable. At high pressures, however, the maximum coefficient of friction is almost independent of pressure, and this is evidence that something other than thermal effects in a Newtonian lubricant is required to explain the experimental observations.

The third important feature of the experimental results is that the curves of coefficient of friction against sliding speed tend towards a limiting 'ceiling' curve for sufficiently high pressures and for sufficiently low temperatures and rolling speeds. This is shown in figure 4, also reproduced from Johnson & Cameron (1967–8); the common limiting curve suggested by figure 4 is also shown in figure 3. Plint (1967–8) reports a similar finding, although his limiting curve is different from that of Johnson & Cameron.

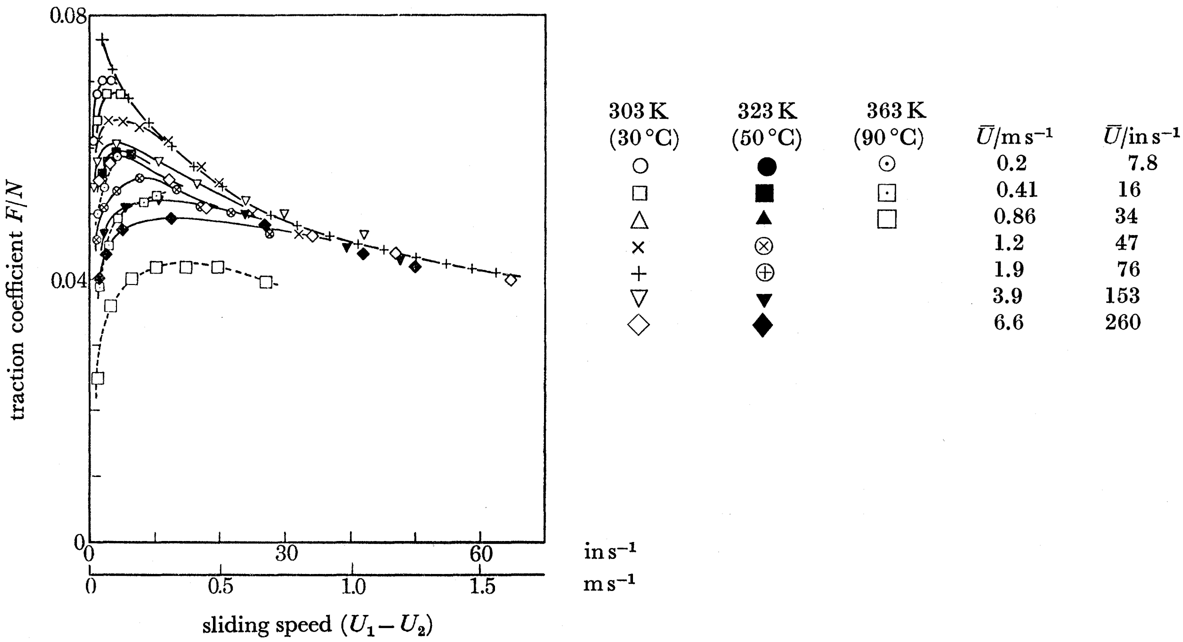


FIGURE 4. Influence of rolling speed and disk temperature on the traction coefficient (from Johnson & Cameron 1967–8).

In the following sections of this paper, an attempt will be made to interpret these features of the experimental results in terms of the viscoelastic properties of the lubricant. The viscoelastic properties of the lubricants used in the friction experiments have not been investigated experimentally, but use will be made of a general model of a real viscoelastic liquid, to which it may be assumed that most lubricants will approximate. The three regions of a typical traction-sliding speed curve, the linear, the nonlinear (ascending), and the thermal (descending) regions will be considered in turn.

The properties of liquids in oscillatory shear at small amplitudes must first be discussed, and this will lead to a consideration of various models which have been proposed to explain this behaviour.

## Properties of liquids in oscillatory shear at small amplitudes

### *Oscillatory shear of a viscoelastic liquid*

The simplest way of describing the properties of a liquid in oscillatory shear at small amplitudes is by its shear modulus, $G^*$, defined as the instantaneous ratio between shear stress and shear strain. In general, the shear stress will have components in phase and out of phase with the shear strain, and $G^*$ is therefore a complex quantity.

$$G^*(\mathrm{i}\omega) = G'(\omega)+\mathrm{i}G''(\omega), \tag{1}$$

where $G'$ is the real part and $G''$ the imaginary part of the modulus. The behaviour in small-amplitude oscillatory shear at any given temperature and pressure is described completely by the variation with frequency of $G'$ and of $G''$.

Alternative descriptions of this behaviour may be given in terms of other complex quantities, e.g. the quantity determined experimentally is the shear mechanical impedance, $Z^*$, defined as the ratio of shear stress to velocity of displacement. The impedance is also a complex quantity, with real and imaginary components:

$$Z^* = Z'+\mathrm{i}Z''.$$

The relation between modulus and impedance is obtained from the standard theory for the propagation of a shear wave through a medium (Barlow & Lamb 1959):

$$(Z^*)^2 = \rho G^*, \tag{2}$$

where $\rho$ is the density of the liquid.

The reciprocals of the above quantities, the compliance, $J^*$, and the admittance, $A^*$, are also important:

$$J^* = J'-\mathrm{i}J'' = (G^*)^{-1}, \tag{3}$$

$$A^* = A'-\mathrm{i}A'' = (Z^*)^{-1}. \tag{4}$$

Equations (2) to (4) give $(A^*)^2 = J^*/\rho$.

Other alternative ways of specifying the properties are in terms of the complex viscosity $\eta^*$ or the complex fluidity $\phi^*$:

$$\eta^* = \eta'-\mathrm{i}\eta'' = G^*/\mathrm{i}\omega,$$

$$\phi^* = \phi'+\mathrm{i}\phi'' = (\eta^*)^{-1} = \mathrm{i}\omega J^*,$$

where $\omega$ is the angular frequency.

### *Models of viscoelastic liquids*

As the time scale of an experiment is decreased, the viscous deformation of the liquid is decreased in proportion, and it is found that the experimental deformation is greater than the theoretical viscous deformation ; for very short times, the behaviour is characteristic more of an elastic solid than of a viscous liquid. It is then postulated that there is an elastic deformation which is in some way added to the viscous deformation to give a liquid with viscoelastic properties. Thus, there is required some way in which a 'softness' property, compliance, admittance, or fluidity, of a viscoelastic fluid may be synthesized from the corresponding properties of a Newtonian fluid and of a Hookean solid.

The conventional way in which this is done is to add the compliances or fluidities of the two elements to form a Maxwell viscoelastic body. Unfortunately, real liquids in general do not behave as Maxwell bodies, and it is usual to explain their behaviour by means of a combination or distribution of Maxwell elements.

This synthesis of a Maxwell element from the compliances of an elastic solid and of a viscous liquid is quite arbitrary, and there is no reason why a viscoelastic element should not be constructed by the addition of the mechanical admittances of the elastic and of the viscous components. Barlow, Lamb *et al.* (1967) pointed out that a large number of chemically defined liquids did in fact behave in this way. They suggested that the factor controlling the variation of viscosity with temperature was of fundamental importance, and that for a liquid to behave as a Maxwell fluid, the viscosity should be limited by the energy of activation, and should show an Arrhenius dependence on temperature:

$$\ln \eta = \alpha_1 + \beta_1/T,$$

where $T$ is the absolute temperature. In practice most liquids would behave like this only near the boiling point. On the other hand, it was suggested that liquids behaving according to the Barlow–Lamb model would have a viscosity limited by free volume and varying with temperature according to the relation:

$$\ln \eta = \alpha_2 + \frac{\beta_2}{T - T_0}.$$

Most practical lubricants would be expected to fulfil these requirements. The Barlow–Lamb model has been shown to fit the experimental results for a wide range of pure chemically defined liquids, including synthetic lubricants such as di(2-ethylhexyl)phthalate. Rather curious deviations have been observed for some mixtures, but there is evidence that the model is obeyed exactly or approximately by mineral oils (Hutton 1968), and in the following treatment it will be assumed that the lubricant behaves as a Barlow–Lamb viscoelastic liquid.

Table 1 gives a summary of the various properties of elastic, viscous and viscoelastic models in oscillatory shear at small amplitudes. The relations defining the viscoelastic models are distinguished by thick boundaries; the other relations follow from these definitions by the rather cumbersome procedure of separating real and imaginary parts in equations such as (1) to (4).

So far the discussion has been concerned with the behaviour of viscoelastic liquids in oscillatory shear, but in elastohydrodynamic lubrication the interest is in their behaviour in continuous shear. At low values of the shear rate $D$, such that

$$\Delta \equiv \eta_0 D / G_\infty \ll 1,$$

the behaviour of a viscoelastic liquid in continuous shear may be determined exactly if its behaviour in oscillatory shear is known. Thus the problem of the apparent viscosity at low sliding speeds should be amenable to exact treatment, but the situation is more difficult at higher shear rates such that $\Delta$ is not small compared with unity. There is then only a rather loose and imprecise analogy between the behaviour of a viscoelastic liquid in oscillatory and in continuous shear, and a corresponding lack of precision must be expected in predictions of the frictional traction. Some of this precision is regained in the thermal region.

TABLE 1. PROPERTIES OF VISCOUS, ELASTIC AND VISCOELASTIC MATERIALS IN OSCILLATORY SHEAR

| property | component | elastic solid | viscous liquid | viscoelastic model liquids: Maxwell | viscoelastic model liquids: Barlow–Lamb |
|---|---|---|---|---|---|
| modulus | $G'$ | $G_\infty$ | 0 | $G_\infty \Omega^2(1+\Omega^2)^{-1}$ | $G_\infty \Omega(1+\Omega^2)^{-2}[\Omega^3-3\Omega+(2\Omega)^{\frac{1}{2}}(1+2\Omega-\Omega^2)]$ |
| | $G''$ | 0 | $\eta_0\omega$ | $G_\infty \Omega(1+\Omega^2)^{-1}$ | $G_\infty \Omega(1+\Omega^2)^{-2}[1-3\Omega^2-(2\Omega)^{\frac{1}{2}}(1-2\Omega-\Omega^2)]$ |
| compliance | $J'$ | $G_\infty^{-1}$ | 0 | $G_\infty^{-1}$ | $G_\infty^{-1}[1+(2/\Omega)^{\frac{1}{2}}]$ |
| | $J''$ | 0 | $(\eta_0\omega)^{-1}$ | $(\eta_0\omega)^{-1}$ | $G_\infty^{-1}[\Omega^{-1}+(2/\Omega)^{\frac{1}{2}}]$ |
| impedance | $Z'$ | $(\rho G_\infty)^{\frac{1}{2}}$ | $(\frac{1}{2}\rho\eta_0\omega)^{\frac{1}{2}}$ | $(\frac{1}{2}\rho G_\infty \Omega)^{\frac{1}{2}}(1+\Omega^2)^{-\frac{1}{2}}[(1+\Omega^2)^{\frac{1}{2}}+\Omega]^{\frac{1}{2}}$ | $(\rho G_\infty)^{\frac{1}{2}}(1+\Omega^2)^{-1}[\Omega^2+(\Omega/2)^{\frac{1}{2}}(1-\Omega)]$ |
| | $Z''$ | 0 | $(\frac{1}{2}\rho\eta_0\omega)^{\frac{1}{2}}$ | $(\frac{1}{2}\rho G_\infty \Omega)^{\frac{1}{2}}(1+\Omega^2)^{-\frac{1}{2}}[(1+\Omega^2)^{\frac{1}{2}}-\Omega]^{\frac{1}{2}}$ | $(\rho G_\infty)^{\frac{1}{2}}(1+\Omega^2)^{-1}[(\Omega/2)^{\frac{1}{2}}(1+\Omega)-\Omega]$ |
| admittance | $A'$ | $(\rho G_\infty)^{-\frac{1}{2}}$ | $(2\rho\eta_0\omega)^{-\frac{1}{2}}$ | $(2\rho G_\infty \Omega)^{-\frac{1}{2}}[(1+\Omega^2)^{\frac{1}{2}}+\Omega]^{\frac{1}{2}}$ | $(\rho G_\infty)^{-\frac{1}{2}}+(2\rho\eta_0\omega)^{-\frac{1}{2}}$ |
| | $A''$ | 0 | $(2\rho\eta_0\omega)^{-\frac{1}{2}}$ | $(2\rho G_\infty \Omega)^{-\frac{1}{2}}[(1+\Omega^2)^{\frac{1}{2}}-\Omega]^{\frac{1}{2}}$ | $(2\rho\eta_0\omega)^{-\frac{1}{2}}$ |
| viscosity | $\eta'$ | 0 | $\eta_0$ | $\eta_0(1+\Omega^2)^{-1}$ | $\eta_0(1+\Omega^2)^{-2}[1-3\Omega^2-(2\Omega)^{\frac{1}{2}}(1-2\Omega-\Omega^2)]$ |
| | $\eta''$ | $G_\infty/\omega$ | 0 | $\eta_0\Omega(1+\Omega^2)^{-1}$ | $\eta_0(1+\Omega^2)^{-2}[\Omega^3-3\Omega+(2\Omega)^{\frac{1}{2}}(1+2\Omega-\Omega^2)]$ |
| fluidity | $\phi'$ | 0 | $\eta_0^{-1}$ | $\eta_0^{-1}$ | $\eta_0^{-1}[1+(2\Omega)^{\frac{1}{2}}]$ |
| | $\phi''$ | $\omega/G_\infty$ | 0 | $\omega/G_\infty$ | $\eta_0^{-1}[\Omega+(2\Omega)^{\frac{1}{2}}]$ |

*Notes*: (1) $\eta_0$ = limiting viscosity at very low frequencies; (2) $G_\infty$ = limiting modulus at very high frequencies; (3) $\Omega = \eta_0\omega/G_\infty$; (4) Positive signs of square roots understood; (5) Expressions defining the two viscoelastic models are enclosed in boxes; the other expressions are derived from them.

## The linear region of the traction-sliding speed curve

### *Non-steady response of a Maxwell liquid*

In an elastohydrodynamic contact such as that of figure 1, the pressure in the film of lubricant gradually builds up from atmospheric, at positions in the inlet zone remote from the conjunction, to something approaching the Hertzian pressure in the conjunction itself. The viscosity varies almost exponentially with pressure, and therefore increases very rapidly as a particle of lubricant enters the Hertzian contact zone.

As an approximation, it will be assumed that lubricant enters the boundary $A_1A_2$ between the inlet and the Hertzian zones in figure 1 with essentially zero viscosity and zero shear stress. As the lubricant crosses the boundary $A_1A_2$, its viscosity will be considered to increase suddenly to a high value $\eta_0$, and to remain at this value until the lubricant leaves the Hertzian zone at $B_1B_2$, when the viscosity returns to zero. The shear rate

$$D = (U_1 - U_2)/h_0$$

is assumed to be constant and small, such that

$$\Delta \equiv D\lambda_M \ll 1,$$

where $\lambda_M = \eta_0/G_\infty$ is the relaxation time of the Maxwell liquid. The equilibrium value of the shear stress $\tau_\infty$ is

$$\tau_\infty = \eta_0 D,$$

but there is a delay in building up this shear stress from its initial value of zero. The build-up of the stress is governed by the constitutive equation of the Maxwell liquid:

$$D = \frac{1}{\eta_0}\left(\tau_M + \lambda_M \frac{d\tau_M}{dt}\right), \tag{5}$$

where $t$ is time. The differential coefficients are unambiguous provided $\Delta$ is small compared with unity. The solution to equation (5) if $\tau_M = 0$ at $t = 0$ is

$$\xi_M \equiv \tau_M/\eta_0 D = 1 - \exp(-\zeta_M),$$

where $\zeta_M = t/\lambda_M$.

The experimental observation is of the mean shear stress $\bar{\tau}_M$ given by

$$\bar{\xi}_M \equiv \frac{\bar{\tau}_M}{\eta_0 D} = \frac{1}{\zeta_{M1}}\int_0^{\zeta_{M1}} \xi_M(\zeta)\, d\zeta = 1 - \frac{1-\exp(-\zeta_{M1})}{\zeta_{M1}}, \tag{6}$$

where $\zeta_{M1} = t_1/\lambda_M$ and $t_1 = 2b/\bar{U}$ is the total time of exposure of the lubricant to the conditions of shear.

Equation (6) has been used both by Crook (1963) and by Plint (1967–8) to explain the variation with rolling speed of the apparent viscosity at low sliding speeds. Plint's values of the shear modulus

$$G_{\infty M} = \eta_0/\lambda_M$$

are of the order of 1 GN $m^{-2}$ ($10^{10}$ dyn $cm^{-2}$) and are thus consistent with estimates from experiments in oscillatory shear (Barlow & Lamb 1959; Hutton 1968), but Crook's values are between 0.14 and 19 MN $m^{-2}$ ($1.4 \times 10^6$ and $1.9 \times 10^8$ dyn $cm^{-2}$) and the lower limit is far too low to be acceptable. It has been shown (Dyson 1965) that these lower values are inconsistent with the observed extent of the linear range of the variation of frictional traction with sliding speed.

*Non-steady response of a Barlow–Lamb liquid*

The relation between the behaviour of a viscoelastic liquid in oscillatory shear and that in continuous shear at low shear rates has been discussed by many authors, e.g. Gross (1953). The only important physical assumption necessary for the development of the theory is that the liquid obeys the Boltzmann principle of linear superposition. A statement of this principle appropriate to the present case is that the shear stress, $\tau(t)$, at a given time, $t$, may be obtained by a linear summation of the fading memories of the strains, $\gamma(u)$, at previous times $u$:

$$\tau(t) = \int_{-\infty}^{t} \frac{d\gamma(u)}{du} \overline{\psi}(t-u)\, du, \tag{7}$$

where $\overline{\psi}$ is the relaxation function. The linearity requirement restricts the application to small strains and to small shear rates.

The strain history considered here is given by:

$$d\gamma(u)/du = 0 \quad (u < 0),$$
$$d\gamma(u)/du = D \quad (u > 0),$$

and equation (7) becomes

$$\tau(t) = D\int_0^t \overline{\psi}(v)\, dv,$$

where $v = t-u$. In terms of the non-dimensional shear stress, $\xi$,

$$\xi = \frac{\tau}{\eta_0 D} = \frac{1}{\eta_0}\int_0^t \overline{\psi}(v)\, dv$$

and the observable mean shear stress is

$$\overline{\xi} = \frac{\overline{\tau}}{\eta_0 D} = \frac{1}{\eta_0 t_1}\int_0^{t_1}\int_0^t \overline{\psi}(v)\, dv\, dt. \tag{8}$$

Thus the behaviour in continuous shear at low shear rates is known if the relaxation function $\overline{\psi}$ is known, and this may be determined from the behaviour of the liquid in oscillatory shear. If

$$\gamma(u) = \gamma_0 \exp(i\omega u)$$

equation (7) becomes

$$\tau(t) = \int_{-\infty}^{t} i\omega\gamma_0 \exp(i\omega u)\, \overline{\psi}(t-u)\, du,$$

and if the variable is again changed to

$$v = t-u,$$

an expression for the instantaneous shear modulus is obtained:

$$G^*(i\omega) = \frac{\tau(t)}{\gamma(t)} = i\omega\int_0^\infty \overline{\psi}(v) \exp(-i\omega v)\, dv. \tag{9}$$

Gross (1953) gives the inversion of this Fourier transform:

$$\overline{\psi}(v) = \frac{2}{\pi}\int_0^\infty \frac{G'(\omega)}{\omega} \sin(\omega v)\, d\omega = \frac{2}{\pi}\int_0^\infty \frac{G''(\omega)}{\omega} \cos(\omega v)\, d\omega,$$

where $G^*(i\omega) = G'(\omega) + iG''(\omega)$.

These integrals are difficult to evaluate as they stand, but they may be put into a more tractable form by the use of the relation between the Fourier and the Laplace transforms. Equation (9) may be put in the form

$$\eta^*(s) = \int_0^\infty \overline{\psi}(v) \exp(-sv)\, dv,$$

where

$$s = \epsilon + i\omega \quad (\epsilon \to 0),$$

and $\eta^*(s) = G^*(s)/s$ is the complex viscosity. This is a Laplace transform, and the inversion is of the form

$$\bar{\psi}(v) = L^{-1}\{\eta^*(s)\}, \tag{10}$$

where the symbol $L^{-1}$ denotes the inverse Laplace transform. Equation (10) has been derived by Phillips (1969). From equation (10), and from a standard result in the theory of Laplace transforms,

$$\int_0^{t_1}\int_0^v [L^{-1}\{f(s)\}]\,dv\,dt = L^{-1}\left\{\frac{1}{s^2}f(s)\right\}$$

the non-dimensional mean shear stress in the situation considered is

$$\bar{\xi} = \frac{1}{\eta_0 t_1}L^{-1}\left\{\frac{G^*(s)}{s^3}\right\}. \tag{11}$$

It is instructive to recalculate the behaviour of a Maxwell liquid by equation (11). For such a liquid

$$G^*_{\mathrm{M}}(s) = G_\infty \frac{\lambda_{\mathrm{M}} s}{1+\lambda_{\mathrm{M}} s},$$

and equation (11) gives

$$\begin{aligned}\bar{\xi}_{\mathrm{M}} &= \frac{G_\infty \lambda_{\mathrm{M}}}{\eta_0 t_1} L^{-1}\left\{\frac{1}{s^2(1+\lambda_{\mathrm{M}} s)}\right\} \\ &= \frac{1}{t_1}L^{-1}\left\{-\frac{\lambda_{\mathrm{M}}}{s}+\frac{1}{s^2}+\frac{\lambda_{\mathrm{M}}^2}{1+\lambda_{\mathrm{M}} s}\right\} \\ &= \frac{1}{t_1}\{-\lambda_{\mathrm{M}}+t_1+\lambda_{\mathrm{M}}\exp(-t_1/\lambda_{\mathrm{M}})\} \\ &= 1-\frac{1-\exp(-\zeta_1)}{\zeta_1},\end{aligned}$$

where $\zeta_1 = t_1/\lambda_{\mathrm{M}}$, in agreement with equation (6). In appendix B, it is shown that a similar treatment applied to a Barlow–Lamb liquid gives the result

$$\bar{\xi}_{\mathrm{BL}} = 1-6(\pi\zeta_1)^{-\frac{1}{2}}+3\zeta_1^{-1}+(2-3\zeta_1^{-1})\exp\zeta_1\,\mathrm{erfc}\,(\zeta_1^{\frac{1}{2}}), \tag{12}$$

where

$$\mathrm{erfc}\,(x) = 2\pi^{-\frac{1}{2}}\int_x^\infty \exp(-x^2)\,dx.$$

The method of solution of this problem was given by Phillips (1969). In figure 5, the development of the mean shear stress is compared for the Maxwell model, equation (6), and for the Barlow–Lamb model, equation (12). For both models, $\bar{\xi}$ tends to $\frac{1}{2}\zeta_1$ at short times and to unity at long times. The main difference between the two models lies in the range of values of $\zeta_1$ in the transition between the behaviour at long times and the behaviour at short times. Thus a change in $\bar{\xi}$ from 0.8 to 0.2 requires about one decade of $\zeta_1$ for the Maxwell liquid and two decades for the Barlow–Lamb liquid. Also the shear stress is reduced by 10 % from its equilibrium value at a non-dimensional time, $\zeta_1$, of 400 for a Barlow–Lamb liquid, compared with 10 for a Maxwell liquid.

This means that viscoelastic effects will be detected at much longer times for a Barlow–Lamb liquid than for a Maxwell liquid with the same relaxation times. However, this predicted difference depends critically on the behaviour of the Barlow–Lamb model at long times in continuous shear, or at low frequencies in oscillatory shear, and some doubt has recently been raised about the applicability of the model in this region (Hutton & Phillips 1969).

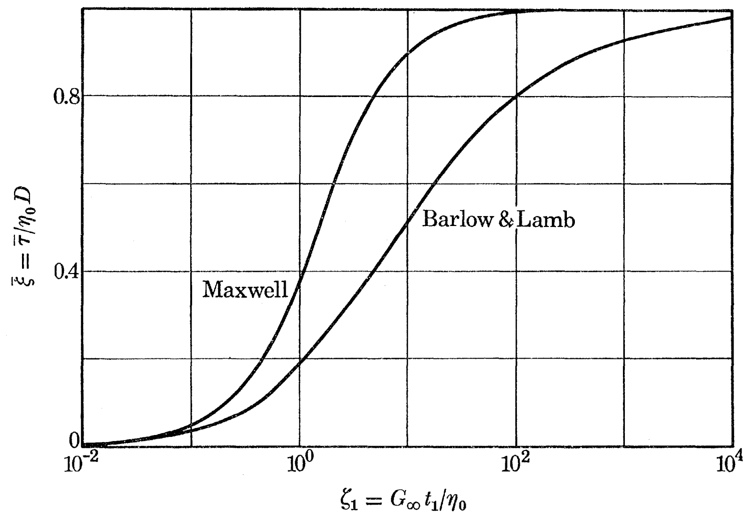


FIGURE 5. Comparison of response of different viscoelastic model liquids in continuous shear at low shear rates.

*Variation of apparent viscosity with rolling speed*

For some purposes it is often more convenient to use the apparent viscosity instead of the shear stress. The apparent viscosity of a liquid in simple shear is defined as

$$\eta_a = \bar{\tau}/D.$$

The variation with rolling speed of the apparent viscosity of a viscoelastic lubricant in elastohydrodynamic lubrication at low sliding speeds may be obtained from the preceding results. Suppose that the true viscosity, $\eta_0$, is constant as the rolling speed, $\bar{U}$, varies. Then the variation with rolling speed of the apparent viscosity $\eta_a$ depends on the variation of the parameter

$$\zeta_1 = \frac{G_\infty t_1}{\eta_0} = \frac{2bG_\infty}{\eta_0 \bar{U}}.$$

The quantities $b$ and $G_\infty$ may be expected to vary over a comparatively narrow range as temperature load, etc., are varied, and most of the variation in $\zeta_1$ will be caused by variations in $\eta_0$. The following values will be assumed:

$$b = 10^{-4}\,\text{m}\ (3.94 \times 10^{-3}\,\text{in}),$$

$$G_\infty = 1\,\text{GN}\,\text{m}^{-2}\ (10^{10}\,\text{dyn}\,\text{cm}^{-2}).$$

It will be seen later that the value of $G_\infty$ chosen is consistent with experimental results in oscillatory shear at high pressures.

Then the variation of the apparent viscosity, $\eta_a$, with the mean rolling speed, $\bar{U}$, may be obtained for various values of the true viscosity, $\eta_0$. The result for a Barlow–Lamb viscoelastic liquid is given in figure 6.

There will be some experimental lower limit to the rolling speed $\bar{U}$; from figure 15 of Johnson & Cameron (1967–8) it seems that this limit is about 0.3 m s$^{-1}$ (12 in s$^{-1}$). This seems reasonable, since it would require a sliding speed of the order of 0.01 m s$^{-1}$ to give a measurable friction, and in the analysis there is an implicit assumption that the sliding speed is small compared with the rolling speed.

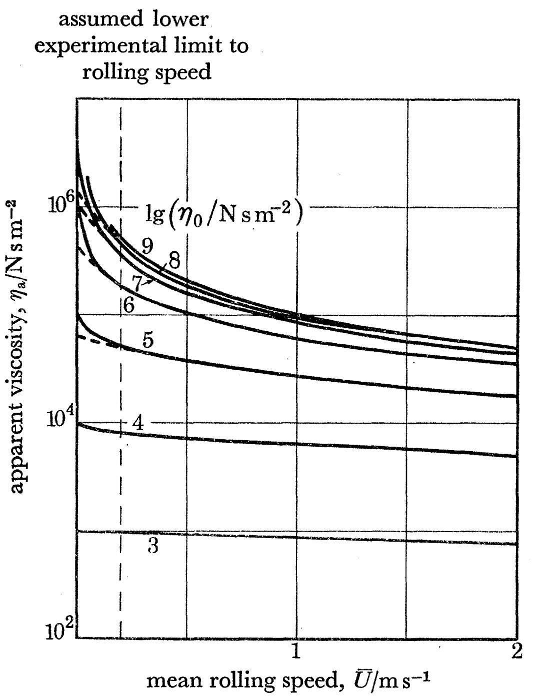


FIGURE 6. Variation of apparent viscosity with rolling speed for Barlow–Lamb viscoelastic liquid. The dashed lines show hypothetical extrapolation to zero rolling speed.

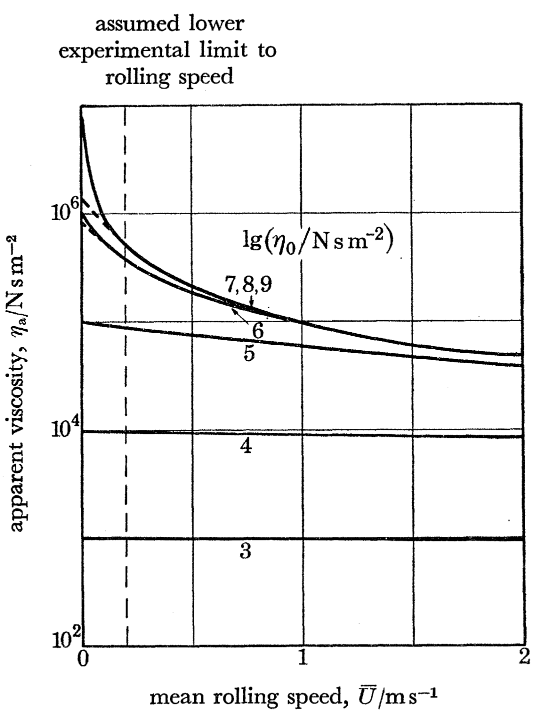


FIGURE 7. Variation of apparent viscosity with rolling speed for Maxwell viscoelastic liquid. The dashed lines again show the hypothetical extrapolation to zero rolling speed.



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In figure 6 the experimental lower limit to the rolling speed has been arbitrarily set at 0.2 m s$^{-1}$ (8 in s$^{-1}$). For low values of the true viscosity, $\eta_0$, e.g. 1 kN s m$^{-2}$ ($10^4$ P), extrapolation of the experimental results to zero rolling speed presents no difficulties but such an extrapolation is much more difficult at high viscosities, e.g. 1 GN s m$^{-2}$ ($10^{10}$ P). The dotted lines show a plausible extrapolation to zero rolling speed of hypothetical sets of experimental readings, limited to a lowest rolling speed of 0.2 m s$^{-1}$. Figure 7 shows the result of similar calculations for a Maxwell model, for which the change in behaviour as the viscosity is increased is more abrupt than for the Barlow–Lamb model.

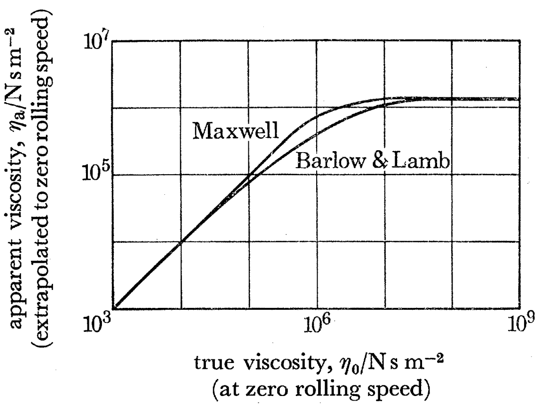


FIGURE 8. Results of extrapolation to zero rolling speed of apparent viscosities of Barlow–Lamb and of Maxwell viscoelastic liquids.

The results of these extrapolations are shown in figure 8, in which the extrapolated viscosities are plotted against the true viscosities. There is a change in the slope of the curve at a viscosity of about 30 kN s m$^{-2}$ ($3 \times 10^5$ P) for the Barlow–Lamb model, but the change does not occur until about 300 kN s m$^{-2}$ ($3 \times 10^6$ P) for the Maxwell model.

The horizontal scale of figure 8 is logarithmic in the true viscosity, and it will therefore be approximately linear in pressure for a liquid obeying the usual viscosity–pressure law. Figure 8

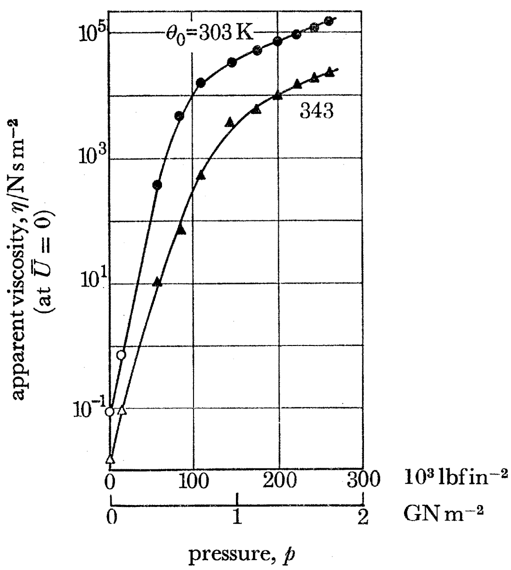


FIGURE 9. The variation of apparent viscosity with pressure and temperature (from Johnson & Cameron 1967–8): ○, △, from static viscosity measurements; ●, ▲, from disk machine measurements.

may therefore be compared with figure 17 of Johnson & Cameron (1967–8), which is reproduced as figure 9 of this paper. This shows the variation with pressure of the apparent viscosity in a disk machine, extrapolated to zero rolling speed.

Some curvature of the lines in figure 9 would be expected, even if the extrapolation to zero rolling speed always gave the true viscosity, $\eta_0$. There seems to be a break in the curves at a viscosity of about 10 kN s m$^{-2}$ ($10^5$ P) and this agrees better with the curve in figure 8 for the Barlow–Lamb liquid than that for the Maxwell liquid. The Maxwell liquid also gives a more abrupt change in slope.

To obtain a break in slope at a viscosity of 30 kN s m$^{-2}$ ($3\times10^5$ P) for a Maxwell liquid, the shear modulus $G_\infty$ would have to be reduced to 0.1 GN m$^{-2}$ ($10^9$ dyn cm$^{-2}$), which is rather low compared with experimental results obtained in oscillatory shear at high pressures.

Since extrapolation of the experimental results to zero rolling speed is so hazardous, the variations of viscosity with rolling speed may be compared by expressing the apparent viscosity as a fraction of its value at some standard rolling speed, e.g. the assumed lower limit of 0.2 m s$^{-1}$. The variation with rolling speed of this ratio is shown in figure 10 for a Barlow–Lamb liquid and in figure 11 for a Maxwellian liquid. Also shown in figures 10 and 11 are the experimental points read from figure 15 of the paper by Johnson & Cameron (1967–8), which is stated to

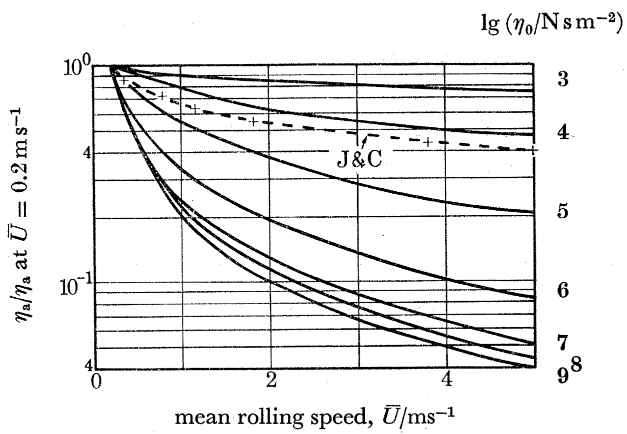


FIGURE 10. Variation of relative apparent viscosity with rolling speed for a Barlow–Lamb liquid. The dashed line (J & C) gives the experimental results of Johnson & Cameron (1967–8).

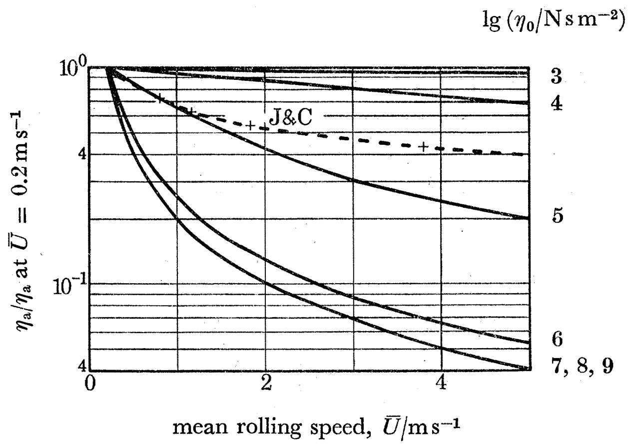


FIGURE 11. Variation of relative apparent viscosity with rolling speed for a Maxwell liquid. The dashed line (J & C) gives the results of Johnson & Cameron again.

be valid for all contact pressures investigated, and therefore presumably valid over a wide range of viscosities. Neither figure 10 nor figure 11 shows good agreement between theory and experiment. The Barlow–Lamb liquid (figure 10) shows a reasonable agreement at a viscosity of about 10 kN s m$^{-2}$($10^5$ P), which is of the right order of magnitude, and agreement is obtained over a rather wider range of viscosities than for the Maxwell liquid (figure 11).

One possible reason for the discrepancy between theory and experiment is the assumption of a single-step function to represent the true transition between the inlet and Hertzian regions of figure 1. The situation is so complicated that it is difficult to deal with it in a more exact manner. Another possible reason for the discrepancy is a change in the pressure distribution as the rolling speed changes. Dowson *et al.* (1962) have shown that, as the rolling speed increases, the elasto-hydrodynamic pressure distribution becomes more sharply peaked. The change will be very slight over the experimental range of rolling speeds, but the viscosity is very sensitive to pressure at high pressures. Thus, the true effective mean viscosity, $\eta_0$, will increase with increasing rolling speed, whereas it was assumed to be constant in the analysis.

A third possible reason for the discrepancy is the deviation of the assumed film thicknesses from the theoretical values for thick films. Johnson & Cameron (1967–8) quote Dyson, Naylor & Wilson (1965–6) to the effect that the theoretical expression of Dowson & Higginson (1961) for the film thickness can be used up to 2.5 μm (100 μin). But figure 10.6 of the paper quoted shows that the theoretical expression is accurate only up to about 1.25 μm (50 μin) for an h.v.i. mineral oil and that at a theoretical film thickness of 2.5 μm (100 μin) the experimental thickness in pure rolling is approximately 1.7 μm. An error in the calculated film thickness would cause the same proportional error in the apparent viscosity.

*Relation between frictional traction and slide/roll ratio*

The relation between the mean frictional shear stress, $\bar{\tau}$, and the sliding speed for a viscoelastic lubricant is given by

$$\bar{\tau} = \eta_0 D \bar{\xi}(\zeta_1), \tag{13}$$

where

$$D = \frac{U_1 - U_2}{h_0}, \quad \zeta_1 = \frac{G_\infty t_1}{\eta_0} = \frac{2bG_\infty}{\eta_0 \bar{U}},$$

and $\bar{\xi}(\zeta_1)$ is given by equation (6) for a Maxwell liquid and by equation (12) for a Barlow–Lamb liquid.

It has been shown experimentally (Dyson *et al.* 1965–6) that the minimum thickness, $h_0$, of film of lubricant is given to a good approximation by the theoretical expression of Dowson & Higginson (1961) over a wide range of experimental conditions. At constant load and temperature, the relation between film thickness and rolling speed is

$$h_0 = C_1 (\bar{U})^{0.7},$$

where $C_1$ is a constant containing viscosity and other fluid properties besides geometrical and elastic constants. Equation (13) may therefore be put in the form

$$\bar{\tau} = C_2 \frac{U_1 - U_2}{\bar{U}} \frac{\bar{\xi}(\zeta_1)}{\zeta_1^{0.3}},$$

where

$$C_2 = \frac{\eta_0^{0.7} (2bG_\infty)^{0.3}}{C_1}.$$

For a given temperature and pressure, $C_2$ will be constant. Since the mean shear stress is proportional to the frictional traction, the coefficient of friction will give a straight line passing through the origin when it is plotted against the slide-roll ratio, $(U_1-U_2)/\bar{U}$. This straight line will be the same for all rolling speeds if the expression $\bar{\xi}(\zeta_1)/\zeta_1^{0.3}$ is approximately constant for varying $\zeta_1$, i.e. for varying rolling speeds. Figure 12 shows that this is true for a Barlow–Lamb liquid over a wider range of $\zeta_1$ than for a Maxwell liquid.

Figure 14 of Johnson & Cameron (1967–8), reproduced here as figure 13 of this paper, shows that the initial slope of the line of the traction coefficient against the slide/roll ratio is, in fact, independent of rolling speed. The theory for the Barlow–Lamb liquid is therefore more nearly consistent with experiment than is that for the Maxwell liquid.

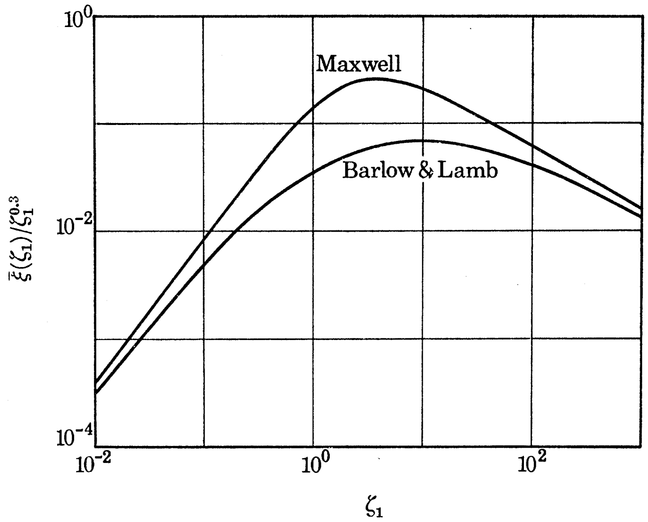


FIGURE 12. Variation of $\bar{\xi}(\zeta_1)/\zeta_1^{0.3}$ with $\zeta_1$ for a Barlow–Lamb liquid and for a Maxwell liquid.

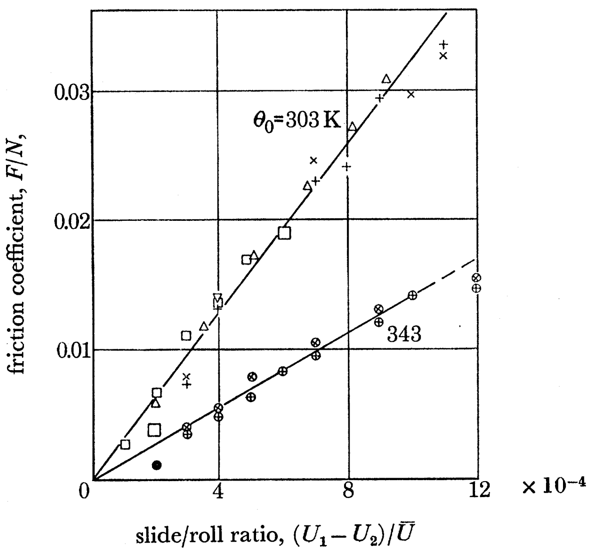


FIGURE 13. Variation of traction coefficient with slide/roll ratio in the isothermal region (from Johnson & Cameron (1967–8)). Symbols denote different rolling speeds.

## The nonlinear (ascending) region of the traction-sliding speed curve

*Relation between behaviour of viscoelastic liquids in oscillatory and in continuous shear*

While the flow properties of a Newtonian fluid are defined by a single property, the viscosity $\eta_0$, those of a viscoelastic fluid require at least one other quantity, e.g. a characteristic time, $\lambda$. In the linear region of the traction-sliding speed curve, which has just been discussed, the shear rate, $D$, is small compared with $\lambda^{-1}$, and the treatment of the rheological properties is relatively simple. In particular, there is a necessary and close connexion between the properties of such a substance in oscillatory and in continuous shear, and this has been used to explain the variations of apparent viscosity with the residence time in the shear field.

In the nonlinear region the shear rate, $D$, is no longer small compared with $\lambda^{-1}$, and this introduces difficulties in the definition of the time derivatives in constitutive equations such as equation (5). These difficulties were pointed out by Fromm (1948) who gave the result for the equilibrium shear stress in a Maxwell viscoelastic fluid:

$$\tau_M = \frac{G_\infty \Delta_M}{1+\Delta_M^2}, \tag{14}$$

where $\Delta_M = D\lambda_M = \eta_0 D/G_{\infty M}$.

This expression is formally identical with that in table 1 for the imaginary component $G''$ of the complex modulus in oscillatory shear, and this result may be generalized to give a general relation between the behaviour of a viscoelastic liquid in oscillatory and in continuous shear

$$\tau(D) = G''(\omega), \tag{15}$$

where $\omega = D$.

The simplicity of this conclusion is attractive, but it is well known that there are three major objections to it:

(i) As the viscosity, $\eta_0$, or the shear rate, $D$, is increased, equations (14) or (15) in general predict that the shear stress rises to a maximum and then decreases. Such behaviour seems improbable on intuitive grounds, and it has been suggested that it may give rise to an instability of of the flow pattern (Fromm 1948; Smith 1960).

(ii) The normal stress distribution given theoretically by a Maxwell material is inconsistent with that given experimentally by real viscoelastic liquids.

(iii) The shear stress becomes comparable with the shear modulus, $G_\infty$, over a wide range of shear rates, and in these circumstances the classical small-strain elasticity theory, which was used in the derivation of equation (14), is invalid.

It is possible that objection (i) may be partly cancelled by objection (iii), as the deviation from classical elasticity theory will be most marked near the maximum in the shear stress. Objection (ii) remains, and it must be concluded that there are factors which intervene in continuous shear at high shear stress, but which do not influence the behaviour in oscillatory shear at small amplitudes. Any necessary connexion between the behaviour in oscillatory and in continuous shear is then lost, but there are special cases in which the connexion may be preserved in a modified form. One modification which is sometimes quoted to equation (15) takes the form

$$\tau(D) = K^{-1}G''(\omega), \tag{16}$$

where $\omega = KD$ and $K$ is an arbitrary shift constant.

In a previous paper (Dyson 1965), the author derived conditions under which equation (16) was valid, using the general constitutive equation for a viscoelastic model liquid proposed by Oldroyd (1958). The conditions given were sufficient but not necessary, and it is possible to derive less restrictive conditions.

*Analysis of results of Smith* (1960)

In the paper previously quoted, the author compared the results of Smith (1960, 1962) and of Crook (1963) in continuous shear of mineral oils in disk machines with those of Barlow & Lamb (1959) in oscillatory shear. Although the oils used in the two classes of experiment were different, it was nevertheless found possible to relate the two sets of results. An example is shown in figure 14,

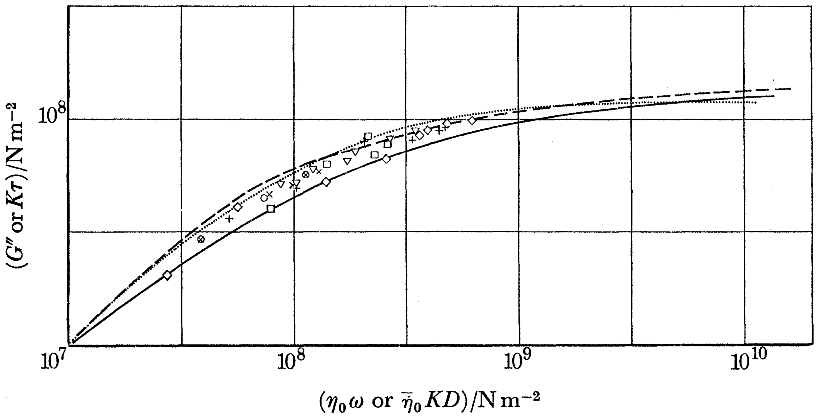


FIGURE 14. Comparison of results of Barlow & Lamb (1959) $G''$ against $\eta_0\omega$, in oscillatory shear (..., l.v.i. mineral oil; ---, m.v.i. mineral oil; ——, h.v.i. mineral oil) with those of Crook (1963), $K\tau$ against $\eta_0 KD$, in continuous shear (+, from figure 22*a* in Crook (1963); ×, ○, □, from figure 22*b*; ▽, from figure 22*c*; ◇, from figure 25).

reproduced from Dyson (1965), where a value of $K$ has been assigned arbitrarily to each combination of experimental conditions, i.e. load, rolling speed, disk temperature. A rather unsatisfactory feature of the treatment was that it was necessary to allow $K$ to vary in an arbitrary manner; it increased with increasing temperature and decreased with increasing pressure. It was also assumed, on the basis of the information then available (Barlow & Lamb 1959) that the shear modulus was independent of pressure and temperature. It now appears that the modulus decreases with increasing temperature (Barlow *et al.* 1967) and increases with increasing pressure (Pursley 1968; Slie & Madigorsky 1968). Qualitatively, this has the same effect as the variation in $K$ which was found necessary to fit the traction results. For the behaviour in oscillatory shear may be described by a relation of the form

$$G''(\omega)/G_\infty = F(\eta_0\omega/G_\infty)$$

and this, in conjunction with equation (16), gives

$$\tau = (G_\infty/K)\,F(\eta_0 KD/G_\infty). \qquad (17)$$

The quantities $K$ and $G_\infty$ appear in equation (17) only as the ratio $G_\infty/K$, and the effects of variations in $K$ are indistinguishable from those of variations in $G_\infty$. Thus, there is the possibility that $K$ may be approximately constant, of the order of unity, as is found experimentally for other viscoelastic liquids.

To test this possibility, it is necessary to consider what is known about the variation of $G_\infty$ with temperature and pressure.

Barlow, Lamb *et al.* (1967) found that there was a linear relation between the temperature and the reciprocal of the shear modulus:

$$\frac{1}{G_\infty} = \frac{1}{G_0} + C(T - T_0).$$

Hutton (1968) found that for an h.v.i. mineral oil at atmospheric pressure

$$\frac{1}{G_\infty} = 2.52 + 0.024\theta,$$

where $G_\infty$ is expressed in units of $\mathrm{GN\,m^{-2}}$ ($10^{10}\,\mathrm{dyn\,cm^{-2}}$) and $\theta$ in deg C. This relation covers the temperature range from 200 to 300 K (−70 to +30 °C) and to use it up to temperatures up to 420 K (150 °C), as it is proposed to do, introduces a gross extrapolation. Nevertheless, it is the only source of information available.

The variation with pressure is even less well explored. It is found that $G_\infty$ increases approximately linearly with pressure at high pressures (Pursely 1968; Slie & Madigorsky 1968); the behaviour at low pressures is difficult to investigate. The experimental results for the slope of the linear relation are given in table 2.

Table 2. Variation of $G_\infty$ with pressure

| liquid | temperature T/K | $\dfrac{dG_\infty}{dp}$ | reference |
|---|---|---|---|
| di(2-ethylhexyl)phthalate | 303 | 2.6 | Pursley 1968 |
| bis-(*m*-(*m*-phenoxy)phenoxy)phenyl ether | 303 | 3.3 | Pursley 1968 |
| glycerol | 257 | 3.3 | Slie & Madigorsky 1968 |

It will be assumed that the variables of pressure and temperature may be separated, i.e. that

$$G_\infty(p,\theta) = G_1(p)\,G_2(\theta) + \text{constant}.$$

Then from Hutton's result

$$\frac{G_2(\theta)}{G_2(20)} = \frac{3}{2.52+0.024\theta}.$$

Since the shear stress depends on $G_\infty/K$ it seems reasonable to seek a relation between $G_\infty/K$ and the quantity

$$\frac{3\bar{p}}{2.52+0.024\theta},$$

where $\bar{p}$ is the mean Hertzian pressure. In figure 15, such a relation is shown for the results of Smith (1960), the values of $K$ being taken from tables 3 and 5 of Dyson (1965), in which it was assumed that $G_\infty$ remained constant, but that $K$ varied. It has been seen that this is indistinguishable from the case in which $G_\infty$ varies but $K$ remains constant, and the constant values to be

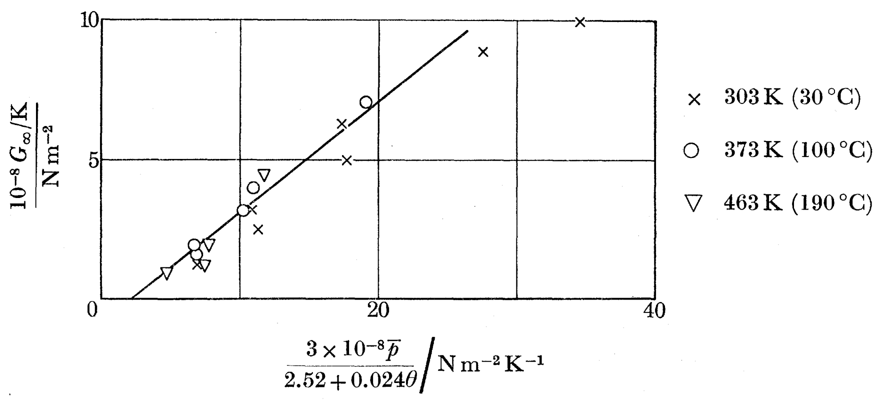


Figure 15. Correlation for $G_\infty/K$ from the results of Smith (1960).

assumed for $G_\infty$ must now be discussed. Barlow & Lamb (1959) gave figures of from 0.7 to 0.8 GN $m^{-2}$ (7 to $8 \times 10^9$ dyn $cm^{-2}$), but these were obtained at temperatures of less than 240 K (−30 °C). From figure 8 of Barlow & Lamb (1959), the range of $(\alpha f)$† of interest is from $10^8$ to $10^{10}$, and from figures 4 to 6 of Barlow & Lamb (1959) the temperatures under these conditions were in the neighbourhood of 273 K (0 °C). It therefore seems better to take Hutton's value of approximately 0.4 GN $m^{-2}$ ($4 \times 10^9$ dyn $cm^{-2}$) at 273 K (0 °C), than the original estimates of 0.7 to 0.8 GN $m^{-2}$ (7 to $8 \times 10^9$ dyn $cm^{-2}$).

Although there is considerable scatter in figure 15, it seems reasonable to conclude that the results of the analysis are consistent with the assumed variation of modulus with temperature, while the variation with pressure will give a slope of 3 at 293 K (20°C) if $K \simeq 7.5$, since the slope of the line drawn in figure 15 is about 0.4. This value for $K$ of 7.5 is now more nearly consistent with experimental results for materials which show viscoelastic behaviour under rather more convenient experimental conditions. Thus a value of $K \simeq 2.0$ has been found for polydimethylsiloxanes (silicone fluids) at atmospheric pressure (Dyson & Wilson 1965–6).

It is therefore possible to conclude that the variation with temperature and pressure of the apparent values of $K$ reported by Dyson (1965) on the assumption that $G_\infty$ was constant, could in fact have been due to variations in $G_\infty$, the value of $K$ being approximately constant at 7.5. It follows that the extrapolation to give $K \simeq 10^3$ at atmospheric pressure, and the interpretation of the results in terms of the variation of the Oldroyd parameter $\mu_0 = \eta_0 K^2$, which were suggested by Dyson (1965), are incorrect.

*Analysis of results of Plint*

In the preceding section, the results of Smith (1960) were analysed in terms of the numerical experimental results obtained from the oscillatory shear of mineral oils. In the present section, the results of Plint (1967–8) will be treated in terms of the analytical model of the Barlow–Lamb liquid.

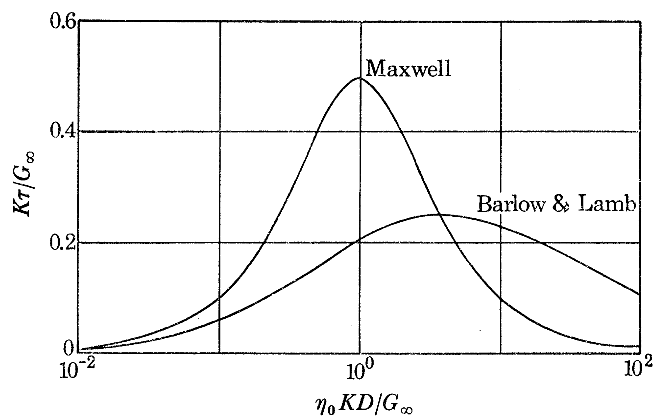


FIGURE 16. Prediction of shear stress for viscoelastic liquids.

Equation (17) suggests that the curve of $(K\tau/G_\infty)$ against $(\eta_0 KD/G_\infty)$ should be identical with that of $(G''/G_\infty)$ against $(\eta_0 \omega/G_\infty)$, and these curves are shown in figure 16 for a Barlow–Lamb and for a Maxwell liquid. The ascending parts of both curves show quite large regions over which the shear stress varies approximately linearly with $\log_{10}(\eta_0 KD/G_\infty)$, and it is tempting to

† In the notation of Barlow & Lamb (1959), $f$ is the frequency (Hz) and $\alpha$ is a shift factor, equal to the ratio of the viscosity at low rates of shear at the test temperature to the value at the reference temperature of 303 K (30 °C).

relate this observation to Plint's finding that the coefficient of friction varies linearly with the logarithm of the sliding speed in this region. This suggested relation may be tested quantitatively by a comparison of the ratios of the slope of the linear portion to the maximum ordinate of the curve. Thus the value of

$$\frac{d(\tau/\tau_{max})}{d\ln(\eta_0 D/G_\infty)}$$

is 0.28 for the Barlow–Lamb liquid and 0.50 for the Maxwell liquid. The results of Plint (1967) for

$$\frac{d(f/f_{max})}{d\ln(U_1-U_2)}$$

range from 0.27 to 0.38 with one exception, and the mean value is 0.35. Here $f$ is the coefficient of friction.

Thus even the simplest version of the viscoelastic analogy gives a slope of the linear portion for a Barlow–Lamb liquid which is within 25% of the experimental value and this agreement is considered quite reasonable in view of the doubtful status of the analogy. The error is rather larger for the Maxwell liquid, and the linear portion of the curve is less extensive than for the Barlow–Lamb liquid.

A number of additional points could be considered in the treatment, e.g.

(*a*) Non-steady shear: the simple analogy between continuous and oscillatory shear holds only in the steady state.

(*b*) The distribution of pressures.

(*c*) Non-linear elasticity.

(*d*) Variations in temperature, both within the oil film and along the bounding surfaces.

(*e*) The normal stress differences.

It is thought that it is not worth while to introduce these refinements at present, because of the lack of quantitative knowledge of the effects and of the approximate nature of the viscoelastic analogy itself.

The discussion in this section has been concerned only with the general shape of the traction-sliding speed curves. An evaluation of the agreement in absolute values depends on a comparison of estimates of the modulus from the two fields of experiment—elastohydrodynamic lubrication and oscillatory shear. This will be discussed in the following section.

## The thermal (descending) region of the traction-sliding speed curve

### *General*

Suppose that the sliding speed in a disk machine is gradually increased from zero, other quantities such as the mean rolling speed, the load, and the bulk disk temperatures being maintained constant. In the linear region the rises in temperature developed within the lubricant film are very small and do not affect the frictional fraction, and this statement is true at the beginning of the nonlinear region. The shear rate is uniform throughout the film, and the relation between shear stress and shear rate is approximately as given by the rising branches of the curves of figure 16.

The curves of figure 16 predict that the shear stress will rise to a maximum and then decrease with increasing sliding speed, even in the absence of thermal effects. It has been argued previously that such behaviour is unlikely on intuitive grounds, and could lead to instability of the flow.

However, suppose the curves of figure 16 flattened out and continued at a constant level for shear rates greater than that corresponding to the maximum in the curves, as illustrated in figure 17. The flow pattern then shows a fairly sudden transition as a result of the decrease in shear modulus with temperature.

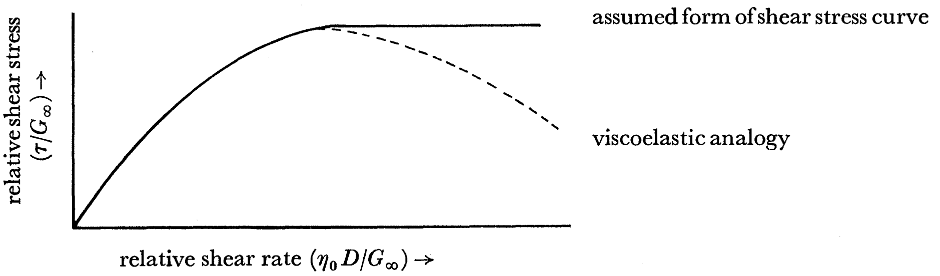


FIGURE 17. Hypothetical form of shear stress/shear rate curve for viscoelastic liquid.

The curves of figure 16 suggest that a sheared viscoelastic liquid is unable to sustain a shear stress greater than a certain fraction of its limiting shear modulus, $G_\infty$, e.g. the $G''$ analogy gives this fraction as one quarter for a Barlow–Lamb liquid. According to heat flow theory (Archard 1958/9) the highest temperature tends to occur in the centre of the film, and the shear modulus will therefore be least at this point. Because of the shape of the curve of figure 17, the shear rate will also be highest in this position, more heat will be generated there, and the temperature will rise still further, The situation is therefore thermally unstable. There will be a fairly sharp transition from a uniform shear field, in which the shear stress or the friction increases with increasing shear rate or sliding speed, to a situation in which most of the shear takes place in a plane near the centre of the film, and the shear stress is controlled by the shear modulus at this position. The shear stress will therefore decrease with increasing sliding speed, since the temperature increases and the shear modulus decreases. It is suggested that the 'shear failure' of Plint (1967–8) should be interpreted in terms of the mechanism just described.

### *Analysis of results of Johnson & Cameron*

The preceding argument may be tested against the experimental results of Johnson & Cameron (1967–8). These authors found that the estimated temperature $\bar{\theta}_c$ in the median plane of the lubricant film was a very important factor in the determination of the frictional traction in the thermal region, and this is qualitatively consistent with the argument just given. It will now be tested quantitatively.

Suppose, as before, that the variables of pressure and temperature may be separated in their effects on the shear modulus, i.e.

$$G_\infty(p,\theta) = G_1(p)\,G_2(\theta) + \text{constant}.$$

Furthermore, suppose $G_2(\theta)$ to be given by Hutton's result for an h.v.i. mineral oil:

$$\frac{G_2(\theta)}{G_2(20)} = \frac{3}{2.52+0.024\theta}.$$

Also suppose that $G_1(p)$ is given by the empirical relation shown in figure 15, which was derived from an analysis of Smith's results in the nonlinear region, given in Dyson (1965). Then the maximum shear stress for a Barlow–Lamb liquid according to the simplest form of the visco-elastic analogy is

$$\tau_{\max} = \frac{G_\infty}{4K} \doteq 0.1\left[\frac{3\bar{p}}{2.52+0.024\theta_c} - 2\times 10^8\right] \tag{18}$$

according to the linear relation suggested in figure 15, since $G''_{\max} = \frac{1}{4}G_\infty$ for a Barlow–Lamb liquid, and the slope of the line in figure 15 is approximately 0.4. In equation (18), $\tau_{\max}$ and $\bar{p}$ are expressed in $\mathrm{N\,m^{-2}}$.

Figure 18 shows a test of equation (18) against the experimental results of Johnson & Cameron (1967–8) as given in figure 20 of their paper. The close agreement is probably largely fortuitous, particularly in view of the fact that different oils were used in the two different investigations. Nevertheless, the agreement indicates that there is probably no gross discontinuity between the nonlinear ascending region of the friction-sliding speed curve and the thermal descending region. It is therefore reasonable to assume that the two regions are governed by the same material properties, and that there is a continuous transition from one region to the other, rather than the discontinuous 'failure' suggested by Plint (1967–8).

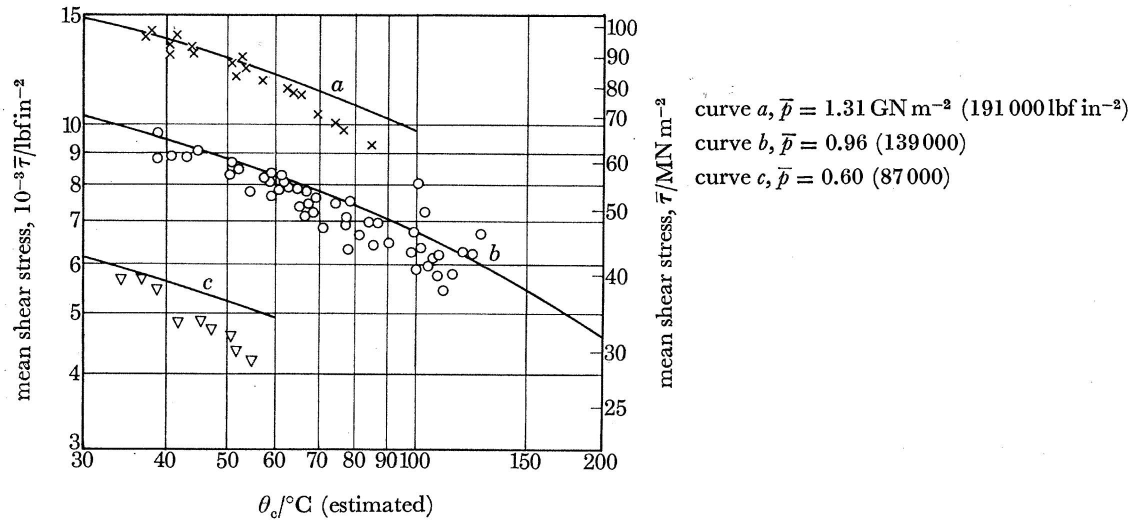


FIGURE 18. Variation of mean shear stress with temperature and pressure. ×, ○, ▽, experimental points from Johnson & Cameron (1967–8); ——, theory based on empirical relation shown in figure 15.

*Variation of friction with disk surface temperature*

Smith (1960, 1962), Johnson & Cameron (1967–8) and Plint (1967–8) found that the frictional traction decreased with increasing temperature of the surfaces of the disks, whereas other workers had reported an effect in the opposite sense (Genkin, Kuz'min & Misharin 1959; Benedict & Kelley 1961, O'Donoghue & Cameron 1966).

It is suggested that the explanation for this discrepancy lies in the fact that the frictional traction goes through a rather flat maximum as the disk surface temperature increases. Such maxima have been observed experimentally in the disk machine at Thornton Research Centre. A theoretical treatment on the basis discussed above is given in appendix C, where it is shown that the temperature corresponding to the maximum friction would be expected to increase with increasing Hertzian pressure, increasing rolling and sliding speed and increasing radius of relative curvature.

## Discussion

*Interpretation of friction results in terms of the properties of a Barlow–Lamb liquid*

The relation between frictional traction and sliding speed in a disk machine has been divided into three regions, the linear region, the nonlinear (ascending) region and the thermal (descending) region. The experimental features of each of these regions have been analysed on the basis of the known or assumed viscoelastic properties of the lubricant.

In the linear region, the slope of the relation between friction and sliding speed may be determined in principle without ambiguity from the viscoelastic properties of the lubricant as determined in oscillatory shear at small amplitudes. In practice, many of the data are uncertain and various approximations have to be used, but the weight of the evidence clearly favours the Barlow Lamb model of a viscoelastic liquid rather than the more usual Maxwell model.

In the nonlinear (ascending) region, the relation between the friction and the viscoelastic properties, as determined in oscillatory shear, is much less clear. However, the tentative correlation which is proposed seems to be consistent with the experimental information available from both fields.

In the thermal (descending) region the relation between friction and the viscoelastic properties as determined in oscillatory shear is again rather closer, but in this case it is necessary to postulate an arbitrary shift constant, $K$. A certain empirical linear relation between $G_\infty/K$ and a simple function of temperature and pressure, shown in figure 15, gives predictions of the friction both in the ascending and descending regions of the curve. A value of $K$ of 7.5 is consistent both with the experimental information on the variation of shear modulus with pressure, and with the behaviour of other viscoelastic liquids, e.g. silicones, in oscillatory and in continuous shear.

*Continuity or discontinuity?*

Arguments have been advanced for a continuity between the ascending and descending regions of the curve of traction against sliding speed, and against the discontinuity suggested by Plint (1967–8). A further argument against Plint's concept of a shear failure, which he likens to that occurring in a plastic solid, is the short duration of exposure of the lubricant to the stress situation, 10 to 100 $\mu$s. Furthermore, the high normal pressures make it impossible for any of the principal stresses in the lubricant to be tensile under physically reasonable circumstances, even when allowance is made for the normal stress differences which occur as a result of the shearing of a viscoelastic liquid. It is difficult to see how failure in the terms described by Plint can occur unless one of the principal stresses is tensile.

*Some unexplained features of Plint's results*

Plint (1967–8) found that the coefficient of friction in the thermal region decreased with increasing dimensions of the conjunction region; his rollers were crowned to give a circular contact zone. It is difficult to explain this relation on the basis of the ideas advanced here, except perhaps on the grounds that the development of the high temperatures in the median plane of the oil film will take a certain time. A small central zone would mean that a small proportion of the area of the median plane of the oil film within the conjunction would be at or near the equilibrium temperature. However, Archard (1958/9) has shown that 95 % of the equilibrium temperature difference is obtained in a time

$$0.3h_0^2/\chi_0,$$

where $\chi_0$ is the thermal diffusivity of the lubricant, about 0.1 $mm^2 s^{-1}$ for mineral oils. For an oil film thickness of 1 $\mu$m, the time required would be approximately 3 $\mu$s, and the corresponding distance would be 3 $\mu$m at a rolling speed of 1 $m s^{-1}$. This is small compared with the dimensions of a conjunction area, and the conclusion seems to be that this mechanism could not account for the observed facts. It is possible that a larger time is required to reach the non-uniform velocity distribution required for equilibrium in the thermal region, but it is difficult to see how a quantitative estimate of this time may be made.

Plint (1967–8) gives a relation between shear stress and temperature, qualitatively similar to, but quantitatively different from, that of Johnson & Cameron (1967–8) reproduced in figure 18. However, Plint's oil had an unusually low viscosity index, and may therefore be expected to differ markedly in composition and in properties from that of Johnson & Cameron. This latter oil had a high viscosity index and was of a type similar to that used by Smith (1960, 1962) and on which the analysis of figure 15 was based. Smith's oil had a rather lower viscosity than that of the one used by Johnson & Cameron, but it would be expected to be otherwise similar in composition and in properties.

*Suggestions for future work*

Experimental work on the frictional traction in elastohydrodynamic lubrication is in progress in many laboratories in different countries, but there are few laboratories with facilities for the investigation of the properties of liquids in oscillatory shear, and even fewer in which this can be done under pressure.

The analysis and interpretation of the results would be simpler if disk machine experiments could be conducted with fluids known to behave according to the simple Barlow–Lamb model, e.g. di(2-ethylhexyl)phthalate. Friction-sliding speed curves could be taken, covering all three regions, as a function of rolling speed, disk surface temperature, load, radius of relative curvature, and the elastic properties of the disks. In principle, if the liquid conforms strictly to the model, the necessary information on the viscoelastic properties of the lubricant in oscillatory shear could be obtained by a measurement of the viscosity and of the real part of the shear mechanical impedance at one frequency, as a function of temperature and of pressure. The accuracy may be insufficient in certain conditions, and it would be useful to have readings over a wide frequency range, both to confirm the validity of the model and to provide a more accurate estimate of the limiting shear modulus at high frequencies.

## Conclusion

The main features of the experimental relation between frictional traction and sliding speed in elastohydrodynamic lubrication have been interpreted in terms of the Barlow–Lamb model of a viscoelastic liquid.

Some features remain unexplained, and there is a need for more information, particularly on the variation with pressure and temperature of the limiting shear modulus, $G_\infty$, in oscillatory shear.

Nevertheless, the results are sufficiently encouraging to suggest that the friction-sliding speed curve represents a continuous relation, controlled by the same viscoelastic properties throughout the whole experimental region.

The author wishes to thank Mr A. Prothero and Mrs A. C. Rowlands for the numerical tabulation of equation (12). He is also indebted to the Council of the Institution of Mechanical Engineers and to Dr K. L. Johnson and Dr R. Cameron for permission to reproduce figures 3, 4, 9, 13 and part of figure 18 of this paper from the *Proceedings* of the Institution, 1967–8.

## Appendix A. Frictional traction for a Newtonian lubricant

Consider the frictional traction in a conjunction such as $A_1B_1A_2B_2$ in figure 1. The boundaries of the film are assumed to be plane and parallel, and all the heat generated by shear is assumed to be removed by conduction through the oil film in a direction perpendicular to the walls. Archard (1958/9) and Crook (1961) have shown that this assumption is adequate under most practical conditions of elastohydrodynamic lubrication. The pressure in the conjunction is assumed to be uniform, and the temperature, $\theta_W$, of the plane bounding surfaces is also assumed to be constant.

The viscosity, $\eta$, of the lubricant is assumed to vary exponentially with temperature:

$$\eta = \eta_W \exp[-\beta(\theta-\theta_W)],$$

where $\eta_W$ is the viscosity at the temperature $\theta_W$. Crook has shown that under these conditions the shear stress, $\tau$, is given by

$$\tau = \frac{\eta_W(U_1-U_2)}{h_0}\,\frac{\ln[(1+\psi)^{\frac{1}{2}}+\psi^{\frac{1}{2}}]}{[\psi(1+\psi)]^{\frac{1}{2}}},$$

where $\psi = \eta_W\beta(U_1-U_2)^2/8k_0$ and $k_0$ is the thermal conductivity of the oil. The variation of shear stress with sliding speed is of interest, and the expression for the shear stress can be put in the form

$$\tau = \frac{1}{h_0}\left(\frac{8k_0\eta_W}{\beta}\right)^{\frac{1}{2}}(1+\psi)^{-\frac{1}{2}}\ln[(1+\psi)^{\frac{1}{2}}+\psi^{\frac{1}{2}}].$$

The function containing $\psi$ has a maximum value of approximately 0.66 when $\psi \simeq 2.3$, and the maximum shear stress is

$$\tau_{max} = \frac{1}{h_0}\left(\frac{8k_0\eta_W}{\beta}\right)^{\frac{1}{2}}0.66.$$

If $\eta_W = \eta_0\exp(\alpha p)$, then the maximum coefficient of friction is given by

$$f_{max} = \frac{\tau_{max}}{p} = \frac{0.66\alpha}{h_0}\left(\frac{8k_0\eta_0}{\beta}\right)^{\frac{1}{2}}\frac{\exp(\frac{1}{2}\alpha p)}{\alpha p}.$$

If $\alpha p \gg 1$, this expression increases rapidly with pressure. A qualitatively similar result is obtained if a Hertzian pressure distribution and a more realistic viscosity–temperature–pressure relation is assumed, and if allowance is made for the variation in surface temperature arising from heat flow to the surfaces.

## Appendix B. Behaviour of a Barlow–Lamb liquid in continuous shear at low shear rates

The expression for the compliance of a Barlow–Lamb liquid in oscillatory shear is

$$J^*(i\omega)\,G_\infty = 1+\left(\frac{2G_\infty}{\eta_0\omega}\right)^{\frac{1}{2}} - i\left[\frac{G_\infty}{\omega\eta_0}+\left(\frac{2G_\infty}{\eta_0\omega}\right)^{\frac{1}{2}}\right].$$

Since

$$(2/i)^{\frac{1}{2}} = \pm(1-i),$$

the expression for the shear modulus may be put in the form

$$G^*(s) = 1/J^*(s) = G_\infty[1+(\lambda s)^{-\frac{1}{2}}]^{-2},$$

where $s = i\omega$ and $\lambda = \eta_0/G_\infty$. Equation (11) of the text then gives the non-dimensional mean shear stress in the situation considered:

$$\bar{\xi}_{BL} = \frac{G_\infty\lambda}{\eta_0 t_1}L^{-1}\left\{\frac{1}{s^2[1+(\lambda s)^{\frac{1}{2}}]^2}\right\}. \tag{B 1}$$

The inverse transform in equation (B 1) is not found in the standard tables, and the nearest related result is

$$L^{-1}\left\{\frac{1}{1+(\lambda s)^{\frac{1}{2}}}\right\} = \frac{1}{(\pi\lambda v)^{\frac{1}{2}}} - \frac{1}{\lambda}\exp(v/\lambda)\,\mathrm{erfc}\,(v/\lambda)^{\frac{1}{2}}, \tag{B 2}$$

where

$$\mathrm{erfc}\,x = \frac{2}{\pi^{\frac{1}{2}}}\int_x^\infty \exp(-x^2)\,\mathrm{d}x.$$

Now

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1}{1+(\lambda s)^{\frac{1}{2}}}\right] = -\frac{\lambda^{\frac{1}{2}}}{2s^{\frac{1}{2}}[1+(\lambda s)^{\frac{1}{2}}]^2}$$

and this, together with the standard result,

$$L^{-1}\left\{\frac{\mathrm{d}}{\mathrm{d}s}f(s)\right\} = -vL^{-1}\{f(s)\},$$

leads to the conclusion that

$$L^{-1}\left\{\frac{1}{s^{\frac{1}{2}}[1+(\lambda s)^{\frac{1}{2}}]^2}\right\} = \frac{2}{\lambda}\left(\frac{v}{\pi}\right)^{\frac{1}{2}} - \frac{2v}{\lambda^{\frac{3}{2}}}\exp\left(\frac{v}{\lambda}\right)\mathrm{erfc}\left(\frac{v}{\lambda}\right)^{\frac{1}{2}}. \tag{B 3}$$

Now let

$$\frac{1}{x^4(1+x)^2} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \frac{E}{1+x} + \frac{F}{x(1+x)^2},$$

where $x$ will later be identified with $(\lambda s)^{\frac{1}{2}}$. The solution is

$$A = -3;\quad B = +3;\quad C = -2;\quad D = 1;\quad E = 3;\quad F = -1,$$

and equation (B 1) now gives

$$\bar{\xi}_{\mathrm{BL}} = \frac{\lambda^3 G_\infty}{\eta_0 t_1} L^{-1}\left\{-\frac{3}{(\lambda s)^{\frac{1}{2}}} + \frac{3}{\lambda s} - \frac{2}{(\lambda s)^{\frac{3}{2}}} + \frac{1}{(\lambda s)^2} + \frac{3}{1+(\lambda s)^{\frac{1}{2}}} - \frac{1}{(\lambda s)^{\frac{1}{2}}[1+(\lambda s)^{\frac{1}{2}}]^2}\right\}.$$

The following results are obtained from standard tables of the Laplace transform:

$$L^{-1}\{s^{-\frac{1}{2}}\} = (\pi v)^{-\frac{1}{2}},$$

$$L^{-1}\{s^{-1}\} = H(v),$$

$$L^{-1}\{s^{-\frac{3}{2}}\} = 2(v/\pi)^{\frac{1}{2}},$$

$$L^{-1}\{s^{-2}\} = v,$$

where $H(v)$ is the Heaviside unit step function, which may be put equal to unity for $v > 0$. The remaining two transforms have already been given. The variable $v$ is equivalent to $t_1$, which may be replaced by the substitution

$$\zeta_1 = t_1/\lambda = G_\infty t_1/\eta_0.$$

The final result is given by equation (12) of the text,

$$\bar{\xi}_{\mathrm{BL}} = 1 - 6(\pi\zeta_1)^{-\frac{1}{2}} + 3\zeta_1^{-1} + (2 - 3\zeta_1^{-1})\exp(\zeta_1)\,\mathrm{erfc}\,(\zeta_1^{\frac{1}{2}}).$$

The expansion of this expression when $\zeta_1 \ll 1$ is

$$\bar{\xi}_{\mathrm{BL}} = \frac{\zeta_1}{2} - \frac{16}{15\pi^{\frac{1}{2}}}\zeta_1^{\frac{3}{2}} + \frac{\zeta_1^2}{2} + \ldots,$$

and when $\zeta_1 \gg 1$ is

$$\bar{\xi}_{\mathrm{BL}} = 1 - \frac{4}{(\pi\zeta_1)^{\frac{1}{2}}} + \frac{3}{\zeta_1} + \ldots.$$

## Appendix C. Relation between frictional traction and disk surface temperature

Let $\theta_1$ be the temperature of the surfaces of the disks entering the conjunction. The mean temperature $\theta_w$ of the surfaces of the disks will be greater than $\theta_1$ because of the 'flash temperature' (Blok 1937). As an approximation, suppose that half the heat generated by friction is removed by each surface and, for the purposes of calculating the flash temperature, let the velocities of each surface relative to their conjunction be put as $\overline{U}$. Then the mean flash temperature is given by

$$\theta_w - \theta_1 = \frac{c\overline{\tau}(U_1-U_2)}{2}\left[\frac{b}{k_m \rho_m c_m \overline{U}}\right]^{\frac{1}{2}}, \tag{C1}$$

where $c$ is a numerical constant, approximately equal to unity; $k_m$ is the thermal conductivity of the materials of the disks; $c_m$ is the specific heat of the materials of the disks; $\rho_m$ is the density of the materials of the disks.

The mean temperature, $\theta_c$, in the median plane of the oil film will be higher than the mean surface temperature, $\theta_w$, because nearly all the heat will be generated by shear near the plane and will then have to be conducted through the oil film to the surfaces (Archard 1958/9)

$$\theta_c - \theta_w = \frac{\tau(U_1-U_2)\,h_0}{4k_0}, \tag{C2}$$

where $h_0$ is the thickness of the film of lubricant.

Now suppose that the shear stress is limited to a constant fraction of the shear modulus, $G_\infty$, of the lubricant, which is related to the local temperature by an equation of the form given by Barlow, Erginsav & Lamb (1967);

$$1/\tau = a_0 + a_1\theta_c \tag{C3}$$

where $a_0$ and $a_1$ are constants. Equations (C1), (C2) and (C3) give

$$\theta_c = \theta_1 + E(a_0 + a_1\theta_c)^{-1}, \tag{C4}$$

where

$$E = (U_1-U_2)\left[\frac{h_0}{4k_0} + (c/2)\left(\frac{b}{k_m\rho_m c_m \overline{U}}\right)^{\frac{1}{2}}\right]. \tag{C5}$$

Now the thickness, $h_0$, of the lubricant film will be given by the relation of Dowson & Higginson (1961)

$$h_0 = 1.6(\eta_0\overline{U})^{0.7}\alpha^{0.6}(E')^{0.03}R^{0.43}w^{-0.13}, \tag{C6}$$

where $\eta_0$ is the viscosity of the lubricant at atmospheric pressure and at the temperature, $\theta_1$, of the surfaces of the disks as they enter the conjunction; $\alpha$ is the pressure coefficient of viscosity; $E'$ is the reduced elastic modulus of the materials of the disks; $R$ is the radius of relative curvature and $w$ is the load per unit transverse width.

The viscosity, $\eta_0$, is related to the temperature, $\theta_1$, e.g. by the Vogel equation:

$$\eta_0 = A\exp[B(\theta_1+\theta_0)^{-1}]. \tag{C7}$$

Equations (C4), (C5), (C6) and (C7) may be put in the dimensionless form

$$\Theta_c = \Theta_1 + \Theta_c^{-1}\{P_1 + P_2\exp[0.7(\Theta_1+P_3)^{-1}]\}, \tag{C8}$$

where

$$\Theta_c = (a_1B)^{-1}(a_0+a_1\theta_c);$$

$$\Theta_1 = (a_1B)^{-1}(a_0+a_1\theta_1);$$

$$P_1 = (2B^2a_1)^{-1}(U_1-U_2)\,cb^{\frac{1}{2}}(k_m\rho_m c_m\overline{U})^{-\frac{1}{2}};$$

$$P_2 = 0.4\,(k_0B^2a_1)^{-1}(U_1-U_2)\,(\overline{U}A)^{0.7}\alpha^{0.6}(E')^{0.03}R^{0.43}w^{-0.13};$$

$$P_3 = (a_1B)^{-1}(a_1\theta_0 - a_0).$$

The variation of $\alpha$ with temperature has been ignored.

The relation between $\Theta_c$ and $\Theta_1$ given in equation (C 8) shows a very flat minimum in $\Theta_c$ with varying $\Theta_1$, and this corresponds to a flat maximum in the relation between the frictional traction and the surface temperature of the disks.

The value of the temperature, $\theta_1$, at which the maximum in the friction occurs may be obtained without difficulty. Differentiation of equation (C 4) gives the condition that $\mathrm{d}\theta_c/\mathrm{d}\theta_1 = 0$ when

$$-\frac{\mathrm{d}E}{\mathrm{d}\theta_1} = a_0 + a_1\theta_c = \frac{1}{\tau} \tag{C 9}$$

from equation (C 3). Now from equation (C 5)

$$\frac{\mathrm{d}E}{\mathrm{d}\theta_1} = \frac{U_1 - U_2}{4k_0}\frac{\mathrm{d}h_0}{\mathrm{d}\theta_1} \tag{C 10}$$

and from equations (C 6) and (C 7)

$$\frac{1}{h_0}\frac{\mathrm{d}h_0}{\mathrm{d}\theta_1} = \frac{0.7}{\eta_0}\frac{\mathrm{d}\eta_0}{\mathrm{d}\theta_1} = -\frac{0.7B}{(\theta_1+\theta_0)^2}. \tag{C 11}$$

Equations (C 9), (C 10) and (C 11) then give

$$\frac{1}{\tau} = \frac{U_1 - U_2}{4k_0}\frac{0.7Bh_0}{(\theta_1+\theta_0)^2},$$

or

$$\frac{\theta_1+\theta_0}{B} \simeq 0.42\left[\frac{\tau h_0(U_1-U_2)}{Bk_0}\right]^{\frac{1}{2}} \tag{C 12}$$

If $\tau = 100\,\mathrm{MN\,m^{-2}}$, $B = 1000\,\mathrm{K}$,

$h_0 = 0.5\,\mu\mathrm{m}$, $k_0 = 0.2\,\mathrm{J\,s^{-1}\,m^{-1}\,K^{-1}}$,

$U_1 - U_2 = 1\,\mathrm{m\,s^{-1}}$, $\theta_0 = -173\,\mathrm{K}$ for $\theta_1$ expressed in K

$\theta_0 = 100\,°\mathrm{C}$ for $\theta_1$ expressed in °C,

then equation (C 12) gives

$$\theta_1 \simeq 381\,\mathrm{K}\,(108\,°\mathrm{C}).$$

Although the shear stress, $\tau$, in equation (C 12) varies only slowly with temperature, the film thickness, $h_0$, varies comparatively rapidly, and the above numerical example is useful only as a guide. Equations (C 6) and (C 7) give

$$h_0 = H\exp\left(\frac{0.7B}{\theta_1+\theta_0}\right),$$

where $H = 1.6\,(A\bar{U})^{0.7}\,\alpha^{0.6}\,(E')^{0.03}\,R^{0.43}\,w^{-0.13}$. Equation (C 12) then gives

$$z\exp(-z^{-1}) \simeq 1.2\left[\frac{\tau H(U_1-U_2)}{Bk_0}\right]^{\frac{1}{2}}, \tag{C 13}$$

where $z = (\theta_1+\theta_0)/0.35B$.

Equation (C 13) may be solved by graphical or numerical interpolation for $\theta_1$, on the assumption that the value of $\tau$ is approximately constant. The results are as stated in the text.

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